# XIX. *On the Solution of Linear Differential Equations. By the Rev.* B. Bronwin. *Communicated by* Samuel Hunter Christie, *Esq., Sec. R.S. &c.*

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If we consider the very different forms which the solutions of Differential Equations differing very little from each other frequently take, and the very different processes often required in each particular case to obtain the solution, we shall be led to conclude that the discovery of any universal or general method of solving them must be a hopeless case. We cannot therefore regard particular methods, especially when applicable to a large number of cases, as useless speculations. The present paper contains the solution of several classes of these equations effected by means of general theorems in the Calculus of Operations adapted to each particular class. For explanation of the symbols employed, let it be observed that D is put for $\frac{d}{dx}$, and that $\varphi$, $\lambda$, $\Psi$, and X denote any functions of $x$, the independent variable, and are the same as $\varphi(x)$, $\lambda(x)$, &c.; and in like manner $\varphi(\mathrm{D})$, $\lambda(\mathrm{D})$, &c. will be used to denote the same functions of D.

## I. FIRST GENERAL THEOREM IN THE CALCULUS OF OPERATIONS.

Let $\tau=\varepsilon^{\int\frac{dx}{\varphi}}$, $\varpi=\varphi\mathrm{D}+\lambda$. We easily verify

$$(\varpi+k)u=\tau^{-k}\varpi\tau^{k}u$$

by substituting for $\tau$ and $\varpi$, and then performing the operations indicated in the result. Change $u$ into $(\varpi+k)u$ in the first member, and into its equal $\tau^{-k}\varpi\tau^{k}u$ in the second, and there results

$$(\varpi+k)^2u=\tau^{-k}\varpi^2\tau^{k}u.$$

A repetition of this process will produce

$$(\varpi+k)^3u=\tau^{-k}\varpi^3\tau^{k}u.$$

In like manner similar equations will be found for higher powers. But the first gives

$$u=(\varpi+k)^{-1}\tau^{-k}\varpi\tau^{k}u.$$

Change $u$ into $\tau^{-k}\varpi^{-1}\tau^{k}u$, and we have

$$\tau^{-k}\varpi^{-1}\tau^{k}u=(\varpi+k)^{-1}u,$$

or transposing,

$$(\varpi+k)^{-1}u=\tau^{-k}\varpi^{-1}\tau^{k}u.$$

Now change $u$ into $(\varpi+k)^{-1}u$ in the first member, and into its equal $\tau^{-k}\varpi^{-1}\tau^{k}u$ in the second, and the result is

$$(\varpi+k)^{-2}u=\tau^{-k}\varpi^{-2}\tau^{k}u\,;$$

and so on. Therefore if $f(\varpi)$ be any function developable in integer powers of $\varpi$, positive or negative, we shall have

$$f(\varpi+k)u=\tau^{-k}f(\varpi)\tau^{k}u,$$

or

$$\tau^{k}f(\varpi+k)u=f(\varpi)\tau^{k}u. \quad \text{(a.)}$$

The assumption $\tau=\varepsilon^{\int\frac{dx}{\varphi}}$ gives $\frac{1}{\varphi}=\frac{\tau'}{\tau}$, and we may regard either $\tau$ or $\varphi$ as the quantity given from which the other is to be found.

*Application of the preceding Theorem to the Solution or Reduction of an extensive class of Linear Differential Equations.*

To abridge we shall put

$$(\varpi+a)(\varpi+a+k)\ldots(\varpi+a+nk)=\mathrm{P}\begin{Bmatrix}a+nk\\ a\end{Bmatrix},$$

$$(\varpi+a)^{-1}(\varpi+a+k)^{-1}\ldots\ldots(\varpi+a+nk)^{-1}=\mathrm{P}\begin{Bmatrix}a+nk\\ a\end{Bmatrix}^{-1};$$

and similarly in other cases. But it is to be observed that we may resolve the last into

$$\frac{\mathrm{A}}{\varpi+a}+\frac{\mathrm{A}_1}{\varpi+a+k}+\ldots\ldots+\frac{\mathrm{A}_n}{\varpi+a+nk},$$

which in practice may be more convenient. The operations implied by the reciprocal factors may be readily performed by a well-known theorem due to Mr. Boole.

Thus

$$(\varpi+m)^{-1}\mathrm{X}=(\varphi\mathrm{D}+\lambda+m)^{-1}\mathrm{X}=\left(\mathrm{D}+\frac{\lambda+m}{\varphi}\right)^{-1}\varphi^{-1}\mathrm{X}$$

$$=\varepsilon^{-\int\frac{\lambda+m}{\varphi}dx}\mathrm{D}^{-1}\varepsilon^{\int\frac{\lambda+m}{\varphi}dx}\varphi^{-1}\mathrm{X}.$$

Now let

$$f(\varpi)\varpi u+pf_{\prime}(\varpi)(\varpi+nk)\tau^{k}u=\mathrm{X}, \quad \text{(1.)}$$

where $f(\varpi)$ and $f_{\prime}(\varpi)$ are any rational functions of $\varpi$, and $k$ may be either positive or negative. To reduce this, assume

$$u=(\varpi+k)(\varpi+2k)\ldots\ldots(\varpi+nk)v.$$

Then by (*a.*)

$$\tau^{k}u=\tau^{k}(\varpi+k)\ldots.(\varpi+nk)v=\varpi(\varpi+k)\ldots.(\varpi+(n-1)k)\tau^{k}v.$$

Substituting these values in (1.), and operating on both members of the result with

$$\varpi^{-1}(\varpi+k)^{-1}\ldots.(\varpi+nk)^{-1}=\mathrm{P}\begin{Bmatrix}nk\\ 0\end{Bmatrix}^{-1},$$

we find

$$f(\varpi)v+pf_{\prime}(\varpi)\tau^{k}v=\mathrm{P}\begin{Bmatrix}nk\\ 0\end{Bmatrix}^{-1}\mathrm{X}, \quad \text{(2.)}$$

which is one order lower than the proposed. All equations therefore of the second order included in (1.) may be considered as integrated by this process.

Another form, distinct from the above, is

$$f(\varpi)(\varpi+nk)u+pf_{\prime}(\varpi)\varpi\tau^{k}u=\mathrm{X}. \quad \ldots \ldots \quad (3.)$$

To reduce this we must assume

$$u=(\varpi+k)^{-1}(\varpi+2k)^{-1}\ldots.(\varpi+nk)^{-1}v.$$

Then by (*a.*) we have

$$\tau^{k}u=\varpi^{-1}(\varpi+k)^{-1}\ldots.(\varpi+(n-1)k)^{-1}\tau^{k}v.$$

Substituting these values and operating with $(\varpi+k)\ldots(\varpi+(n-1)k)$ on both members of the result, we have

$$f(\varpi)v+pf_{\prime}(\varpi)\tau^{k}v=\mathrm{P}\begin{Bmatrix}(n-1)k\\ k\end{Bmatrix}\mathrm{X}. \quad \ldots \ldots \quad (4.)$$

The observations made with reference to the former example might be repeated here. I shall add, that as $k$ may be both positive and negative, these two examples include every variety of case of this form of equation.

Before I proceed to notice particular examples, it may be as well to give a few more general ones, and thus to point out the whole series of them which are susceptible of reduction by this method.

Let $$f(\varpi)\varpi(\varpi+k)u+pf_{\prime}(\varpi)(\varpi+nk)\tau^{2k}u=\mathrm{X}. \quad \ldots \ldots \quad (5.)$$

Make $$u=(\varpi+2k)(\varpi+3k)\ldots..(\varpi+(n+1)k)v,$$

the common difference of the factors being $k$, as before, which here also may be both positive and negative. This will give

$$\tau^{2k}u=\varpi(\varpi+k)\ldots..(\varpi+(n-1)k)\tau^{2k}v,$$

and proceeding exactly as before, we shall arrive at the reduced equation,

$$f(\varpi)(\varpi+(n+1)k)v+pf_{\prime}(\varpi)\tau^{2k}v=\mathrm{P}\begin{Bmatrix}nk\\ 0\end{Bmatrix}^{-1}\mathrm{X}. \quad \ldots \ldots \quad (6.)$$

Again, suppose

$$f(\varpi)(\varpi+nk)(\varpi+nk+k)u+pf_{\prime}(\varpi)\varpi\tau^{2k}u=\mathrm{X}. \quad \ldots \ldots \quad (7.)$$

The assumption

$$u=(\varpi+2k)^{-1}(\varpi+3k)^{-1}\ldots.(\varpi+(n+1)k)^{-1}v$$

leads to

$$f(\varpi)(\varpi+k)v+pf_{\prime}(\varpi)\tau^{2k}v=\mathrm{P}\begin{Bmatrix}(n-1)k\\ k\end{Bmatrix}\mathrm{X}. \quad \ldots \ldots \quad (8.)$$

The two last examples, like the former, are reduced an order lower; and when they are of the second order, they may be considered as integrated. In order to enable us to effect their reduction, it is necessary that we should have two operating factors, as $\varpi(\varpi+k)$, in one of the terms, these factors having the difference $k$. I shall only give two more examples, which will suffice to indicate the series of them before mentioned.

Let $$f(\varpi)\varpi(\varpi+k)(\varpi+2k)u+pf_{\prime}(\varpi)(\varpi+nk)\tau^{3k}u=\text{X},$$
and
$$f(\varpi)(\varpi+nk)(\varpi+nk+k)(\varpi+nk+2k)u+pf_{\prime}(\varpi)\varpi\tau^{3k}u=\text{X}.$$

The first of these, by making
$$u=(\varpi+3k)(\varpi+4k)\dots(\varpi+(n+2)k)v,$$
will reduce to
$$f(\varpi)(\varpi+nk+k)(\varpi+nk+2k)v+p\tau^{3k}v=\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X},$$
and the second, by putting
$$u=(\varpi+3k)^{-1}\dots(\varpi+(n+2)k)^{-1}v,$$
reduces to
$$f(\varpi)(\varpi+k)(\varpi+2k)v+p\tau^{3k}v=\text{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\text{X}.$$

It will be observed that the equations in the two last examples are of the third order at the lowest, and those to which they are reduced of the second; and if we were to continue the series, they would rise an order at every step. But we will here leave them and proceed to give a few particular examples.

In (1.) and (2.) make
$$f(\varpi)=\varpi+a,\ f_{\prime}(\varpi)=1\ ;$$
and they become
$$\varpi(\varpi+a)u+p(\varpi+nk)\tau^{k}u=\text{X} \quad \dots\dots\dots \quad (9.)$$
$$(\varpi+a)v+p\tau^{k}v=\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X}.$$
The last gives
$$v=(\varpi+a+p\tau^{k})^{-1}\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X},$$
and therefore
$$u=\text{P}\begin{Bmatrix}nk\\k\end{Bmatrix}(\varpi+a+p\tau^{k})^{-1}\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X}.$$

It is not necessary to reduce the value of $u$ any further, as the mode of doing it has been made sufficiently plain, and moreover it is quite as convenient as it stands. We shall only observe that if, to abridge, we make
$$\frac{\lambda+a+p\tau^{k}}{\varphi}=\Phi,$$
we have
$$(\varpi+a+p\tau^{k})^{-1}=(\text{D}+\Phi)^{-1}\varphi^{-1}=\varepsilon^{-\int\Phi dx}\text{D}^{-1}\varepsilon^{\int\Phi dx}\varphi^{-1}.$$

If we wish to see (9.) under the ordinary form, it is easily reduced, first to
$$\varpi^{2}u+(a+p\tau^{k})\varpi u+(n+1)kp\tau^{k}u=\text{X};$$
and then by substituting for $\varpi$ to
$$\varphi^{2}\text{D}^{2}u+\varphi(\varphi'+2\lambda+a+p\tau^{k})\text{D}u+(\varphi\lambda'+\lambda^{2}+(a+p\tau^{k})\lambda+(n+1)kp\tau^{k})u=\text{X}.$$

If we would deduce particular integrable equations from this, we may assume $\tau$ and $\lambda$ any functions of $x$ at pleasure, and the relation $\frac{1}{\varphi}=\frac{\tau'}{\tau}$ will give $\varphi$. It will be observed that the accent upon $\tau$, $\lambda$, and $\phi$ is employed to denote differential coefficients. But as this equation, and many more which may be deduced, contain two arbitrary functions of the independent variable, the number of particular practicable forms is immense. To select examples therefore would be very difficult.

Now make in (1.) and (2.)

$$f(\varpi)=\varpi+a,\ f_{\prime}(\varpi)=\varpi+b,$$

and we have

$$\varpi(\varpi+a)u+p(\varpi+b)(\varpi+nk)\tau^k u=\mathrm{X}\ .\ .\ .\ .\ .\ .\ .\ .\ (10.)$$

$$(\varpi+a)v+p(\varpi+b)\tau^k v=\mathrm{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X},$$

or

$$(\varpi+a)v+p\tau^k(\varpi+b+k)v=\mathrm{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X},$$

and therefore

$$(1+p\tau^k)\varpi v+(a+(b+k)p\tau^k)v=\mathrm{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X}.$$

To abridge make

$$\frac{a+(b+k)p\tau^k}{1+p\tau^k}=\Phi,$$

and the last equation becomes

$$(1+p\tau^k)(\varpi+\Phi)v=\mathrm{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X};$$

whence

$$v=(\varpi+\Phi)^{-1}(1+p\tau^k)^{-1}\mathrm{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X},\ u=\mathrm{P}\begin{Bmatrix}nk\\k\end{Bmatrix}(\varpi+\Phi)^{-1}(1+p\tau^k)^{-1}\mathrm{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X}.$$

Similarly, from (3.) and (4.) we derive

$$(\varpi+a)(\varpi+nk)u+p\varpi\tau^k u=\mathrm{X},\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (11.)$$

where

$$u=\mathrm{P}\begin{Bmatrix}nk\\k\end{Bmatrix}^{-1}(\varpi+a+p\tau^k)^{-1}\mathrm{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\mathrm{X}.$$

The last equation, reduced to the ordinary form, making $c=a+nk$, becomes

$$\phi^2\mathrm{D}^2u+\phi(\phi'+2\lambda+c+p\tau^k)\mathrm{D}u+(\phi\lambda'+\lambda^2+(c+p\tau^k)\lambda+nak+kp\tau^k)u=\mathrm{X}.$$

From (3.) and (4.) we also deduce

$$(\varpi+a)(\varpi+nk)u+p\varpi(\varpi+b)\tau^k u=\mathrm{X}\ .\ .\ .\ .\ .\ .\ .\ .\ (12.)$$

$$(\varpi+a)v+p(\varpi+b)\tau^k v=\mathrm{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\mathrm{X}.$$

From these, exactly as in (10.), we find

$$u=\mathrm{P}\begin{Bmatrix}nk\\k\end{Bmatrix}^{-1}(\varpi+\Phi)^{-1}(1+p\tau^k)^{-1}\mathrm{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\mathrm{X},$$

where $\Phi$ has the same value as in (10.).

The examples (10.) and (12.), if reduced to the ordinary form, would be very different from those which have been so reduced, and they would be considerably more complex. And we may observe that if the functions $f(\varpi)$, $f_{\prime}(\varpi)$ in (1.), (3.), &c. have suitable factors, the reduced equations (2.), (4.), &c. may be still further reduced by the same means, the proposed equations in this case being of an order above the second.

From (5.) and (6.), making $f(\varpi)=f_{\prime}(\varpi)=1$, we have

$$\varpi(\varpi+k)u+p(\varpi+nk)\tau^{2k}u=\text{X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

and

$$(\varpi+(n+1)k)v+p\tau^{2k}v=\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X};$$

whence

$$v=\{\varpi+(n+1)k+p\tau^{2k}\}^{-1}\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X},$$

$$u=\text{P}\begin{Bmatrix}(n+1)k\\2k\end{Bmatrix}\{\varpi+(n+1)k+p\tau^{2k}\}^{-1}\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X}.$$

Similarly, from (7.) and (8.) we find

$$(\varpi+nk)(\varpi+nk+k)u+p\varpi\tau^{2k}u=\text{X}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

$$v=\{\varpi+k+p\tau^{2k}\}^{-1}\text{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\text{X},$$

$$u=\text{P}\begin{Bmatrix}(n+1)k\\2k\end{Bmatrix}^{-1}\{\varpi+k+p\tau^{2k}\}^{-1}\text{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\text{X}.$$

The two last examples, reduced to the ordinary form, are

$$\phi^2\text{D}^2u+\phi(\phi'+2\lambda+k+p\tau^{2k})\text{D}u+(\phi\lambda'+\lambda^2+\lambda(k+p\tau^{2k})+(n+2)kp\tau^{2k})u=\text{X},$$

and

$$\phi^2\text{D}^2u+\phi(\phi'+2\lambda+(2n+1)k+p\tau^{2k})\text{D}u+(\phi\lambda'+\lambda^2+(2n+1)k\lambda+(2k+\lambda)p\tau^{2k}+n(n+1)k^2)u=\text{X}$$

respectively.

I shall only give two other examples, derived from the same source with the two preceding.

$$\varpi(\varpi+k)u+p(\varpi+a)(\varpi+nk)\tau^{2k}u=\text{X},$$

$$u=\text{P}\begin{Bmatrix}(n+1)k\\2k\end{Bmatrix}(\varpi+\Psi)^{-1}(1+p\tau^{2k})^{-1}\text{P}\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\text{X}$$

$$(\varpi+nk)(\varpi+nk+k)u+p\varpi(\varpi+a)\tau^{2k}u=\text{X},$$

$$u=\text{P}\begin{Bmatrix}(n+1)k\\2k\end{Bmatrix}^{-1}(\varpi+\Phi)^{-1}(1+p\tau^{2k})^{-1}\text{P}\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\text{X};$$

where

$$\Psi=\frac{(n+1)k+(a+2k)p\tau^{2k}}{1+p\tau^{2k}},\quad \Phi=\frac{k+(a+2k)p\tau^{2k}}{1+p\tau^{2k}}.$$

## II. SECOND GENERAL THEOREM IN THE CALCULUS OF OPERATIONS.

Make $\text{D}=\text{D}_1+\text{D}_2$, $\text{D}_1$ operating upon $u$ only and $\text{D}_2$ upon $x$ only. Then by Taylor's theorem

$$f(\text{D})=f(\text{D}_1+\text{D}_2)=f(\text{D}_1)+\text{D}_2f'(\text{D}_1)+\tfrac{1}{2}\text{D}_2^2f''(\text{D}_1)+\ldots.$$

This formula will be of frequent use in the subsequent part of this paper. By it

$$\varepsilon^{kf(\mathrm{D})}=\varepsilon^{kf(\mathrm{D}_1)+k\mathrm{D}_2f'(\mathrm{D}_1)+\dots}=\varepsilon^{kf(\mathrm{D}_1)}\varepsilon^{k\mathrm{D}_2f'(\mathrm{D}_1)+}=\{1+k\mathrm{D}_2f'(\mathrm{D}_1)+\}\varepsilon^{kf(\mathrm{D}_1)}.$$

Operate with both members of this upon $\{\phi(\mathrm{D})x+\lambda(\mathrm{D})\}u$, and we have

$$\varepsilon^{kf(\mathrm{D})}\{\phi(\mathrm{D})x+\lambda(\mathrm{D})\}u=\{\phi(\mathrm{D})x+\lambda(\mathrm{D})+k\phi(\mathrm{D})f'(\mathrm{D})\}\varepsilon^{kf(\mathrm{D})}u,$$

dropping the mark under D, as being no further needed, D everywhere now operating upon all that follows it. Make

$$f(\mathrm{D})=\int\frac{d\mathrm{D}}{\phi(\mathrm{D})},$$

then

$$f'(\mathrm{D})=\frac{1}{\phi(\mathrm{D})},$$

and the last equation becomes

$$\varepsilon^{k\int\frac{d\mathrm{D}}{\phi(\mathrm{D})}}\Big\{\phi(\mathrm{D})x+\lambda(\mathrm{D})\Big\}u=\Big\{\{\phi(\mathrm{D})x+\lambda(\mathrm{D})+k\Big\}\varepsilon^{k\int\frac{d\mathrm{D}}{\phi(\mathrm{D})}}u.$$

Now let

$$\phi(\mathrm{D})x+\lambda(\mathrm{D})=\varpi',\ \varepsilon^{\int\frac{d\mathrm{D}}{\phi(\mathrm{D})}}=\tau',$$

which gives

$$\frac{1}{\phi(\mathrm{D})}=\frac{1}{\tau'}\,\frac{d\tau'}{d\mathrm{D}};$$

and the preceding becomes

$$\tau'^k\varpi'u=(\varpi'+k)\tau'^ku.$$

Change $u$ into $\tau'^{-k}u$, and there results

$$\tau'^k\varpi'\tau'^{-k}u=(\varpi'+k)u,$$

or

$$(\varpi'+k)u=\tau'^k\varpi'\tau'^{-k}u$$

by transposing the members.

From the last equation, by the same course of reasoning by which (*a.*) was established, we find

$$\tau'^{-k}f(\varpi'+k)u=f(\varpi')\tau'^{-k}u. \quad \dots\dots\dots\dots (b.)$$

If in (*a.*) we change $x$ into D and D into $x$, and also $\varepsilon$ into $\varepsilon^{-1}$, we convert (*a.*) into (*b.*); and the same conversion of symbols will change (*b.*) into (*a.*).

*Application of the preceding Theorem.*

The equations

$$f(\varpi')\varpi'u+pf_{\prime}(\varpi')(\varpi'+nk)\tau'^{-k}u=\mathrm{X},$$
$$f(\varpi')(\varpi'+nk)u+pf_{\prime}(\varpi')\varpi'\tau'^{-k}u=\mathrm{X},$$
$$f(\varpi')\varpi'(\varpi'+k)u+pf_{\prime}(\varpi')(\varpi'+nk)\tau'^{-2k}u=\mathrm{X},\ \&c.$$

by the assumptions

$$u=(\varpi'+k)\dots(\varpi'+nk)v,$$
$$u=(\varpi'+k)^{-1}\dots(\varpi'+nk)^{-1}v,$$
$$u=(\varpi'+2k)\dots(\varpi'+(n+1)k)v,\ \&c.$$

will reduce to

$$f(\varpi')v+pf_{\prime}(\varpi')\tau'^{-k}v=\mathrm{P}'\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X},$$

$$f(\varpi')v+pf_{\prime}(\varpi')\tau'^{-k}v=\mathrm{P}'\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}\mathrm{X},$$

$$f(\varpi')(\varpi'+(n+1)k)v+pf_{\prime}(\varpi')\tau'^{-2k}v=\mathrm{P}'\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}\mathrm{X},\ \&c.$$

respectively, the reductions being made by the theorem (*b*.).

But these equations and every step of their reduction may be derived immediately from (1.), (3.), &c., and their reductions by the conversion of symbols before mentioned, which is not a little remarkable. Hence we may derive the solutions of the one series from those of the other merely by the interchange of symbols.

In the equations of which we are now treating, $\varphi(\mathrm{D})$ and $\lambda(\mathrm{D})$ must be rational functions of D; and we must also have $\tau'^{-k}=\chi(\mathrm{D})$, a rational function. Therefore we must have

$$\frac{1}{\varphi(\mathrm{D})}=-\frac{\chi'(\mathrm{D})}{k\chi(\mathrm{D})},$$

and

$$\varphi(\mathrm{D})=-\frac{k\chi(\mathrm{D})}{\chi'(\mathrm{D})}.$$

In the values of $u$ we have operating factors of the form

$$(\varpi'+rk)^{-1}=(\varphi(\mathrm{D})x+\lambda(\mathrm{D})+rk)^{-1}=\left(x+\frac{\lambda(\mathrm{D})+rk}{\varphi(\mathrm{D})}\right)^{-1}\varphi(\mathrm{D})^{-1}=\varepsilon^{\int\frac{\lambda(\mathrm{D})+rk}{\varphi(\mathrm{D})}d\mathrm{D}}x^{-1}\varepsilon^{-\int\frac{\lambda(\mathrm{D})+rk}{\varphi(\mathrm{D})}d\mathrm{D}}\varphi(\mathrm{D})^{-1}$$

by a well-known theorem due to Mr. BOOLE. Now in order that these may be practicable in finite terms, or as it is usual to say, capable of interpretation, we must have

$$\varepsilon^{\int\frac{\lambda(\mathrm{D})}{\varphi(\mathrm{D})}d\mathrm{D}}=\Phi(\mathrm{D})\varepsilon^{m\mathrm{D}},$$

$\Phi(\mathrm{D})$ being a rational function of D, and $m$ being any constant, positive or negative, or nothing. This is the most general form possible, and it gives

$$\frac{\lambda(\mathrm{D})}{\varphi(\mathrm{D})}=\frac{m\Phi(\mathrm{D})+\Phi'(\mathrm{D})}{\Phi(\mathrm{D})},$$

and

$$\lambda(\mathrm{D})=\frac{\varphi(\mathrm{D})}{\Phi(\mathrm{D})}(m\Phi(\mathrm{D})+\Phi'(\mathrm{D})).$$

The expression $\varepsilon^{\int\frac{rk}{\varphi(\mathrm{D})}d\mathrm{D}}$ has been rendered practicable in making $\tau'^{-k}$ so.

The only other operating factor which we have to consider is

$$(\varpi'+h+p\tau'^{-k})^{-1},$$

$h$ being some constant. Now putting for $\tau'^{-k}$ its value before found, this will reduce to

$$\left\{x+\frac{\lambda(\mathrm{D})+h+p\chi(\mathrm{D})}{\varphi(\mathrm{D})}\right\}^{-1}\varphi(\mathrm{D})^{-1}.$$

The exponentials depending on $\frac{\lambda(\mathrm{D})}{\varphi(\mathrm{D})}$ and $\frac{h}{\varphi(\mathrm{D})}$ have been considered, and it only remains to consider

$$\varepsilon^{p\int\frac{\chi(D)}{\phi(D)}dD}=\varepsilon^{-\frac{p}{k}\int\chi'(D)dD}=\varepsilon^{-\frac{p}{k}\chi(D)}$$

by substituting for $\frac{1}{\phi(D)}$ its value. But $\varepsilon^{-\frac{p}{k}\chi(D)}$ cannot be interpreted except when $\chi(D)=qD$. We have therefore, finally,

$$\tau'^{-k}=qD,\ \phi(D)=-kD,\ \lambda(D)=-\frac{kD}{\Phi(D)}(m\Phi(D)+\Phi'(D)).$$

This result narrows very much the limits of the practicable cases.

We may however obtain a small increase of generality by making $\chi(D)=qD+r$. There is nothing to prevent this, since

$$\varepsilon^{-\frac{p}{k}(qD+r)}=\varepsilon^{-\frac{pr}{k}}\varepsilon^{-\frac{pq}{k}D},$$

which makes no change in the interpretation of the results, the only difference which it occasions being the introduction of the constant quantity $\varepsilon^{-\frac{pr}{k}}$ into the value of $u$ where we should otherwise have unity in its place. Thus we shall have

$$\tau'^{-k}=qD+r,\ \phi(D)=-k\left(D+\frac{r}{q}\right),\ \lambda(D)=-k\left(D+\frac{r}{q}\right)\left(m+\frac{\Phi'(D)}{\Phi(D)}\right);$$

which values give

$$\varpi'=-k\left(D+\frac{r}{q}\right)\left(x+m+\frac{\Phi'(D)}{\Phi(D)}\right).$$

By the conversion of symbols we derive from (9.),

$$\varpi'(\varpi'+a)u+p(\varpi'+nk)\tau'^{-k}u=X,$$

$$u=P'\begin{Bmatrix}nk\\k\end{Bmatrix}(\varpi'+a+p\tau'^{-k})^{-1}P'\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}X,$$

which by putting for $\tau'^{-k}$ its value becomes

$$\varpi'(\varpi'+a)u+p(\varpi'+nk)(qD+r)u=X \quad \text{. . . . . . . . . . (15.)}$$

$$u=P'\begin{Bmatrix}nk\\k\end{Bmatrix}(\varpi'+pqD+pr+a)^{-1}P'\begin{Bmatrix}nk\\0\end{Bmatrix}^{-1}X.$$

From (11.) we derive in like manner

$$(\varpi'+a)(\varpi'+nk)u+p\varpi'(qD+r)u=X \quad \text{. . . . . . . . (16.)}$$

$$u=P'\begin{Bmatrix}nk\\k\end{Bmatrix}^{-1}(\varpi'+pqD+pr+a)^{-1}P'\begin{Bmatrix}(n-1)k\\k\end{Bmatrix}X.$$

But (10.) and (12.) treated thus would lead to equations of the third order; and as we cannot notice those of all orders, we shall pass these by.

It would render the values of $u$ too long and complex to substitute for $\varpi'$ its value and reduce them further. But this is unnecessary, since the method of doing it has been made sufficiently plain, and indeed is well understood. If we would reduce these equations to the ordinary form, it can easily be effected by the formula

$$f(D)=f(D_1+D_2)=f(D_1)+D_2f'(D_1)+\frac{1}{2}D_2^2f''(D)+\&c.,$$

where $D_1$ operates upon $u$, $D_2$ upon $x$. Thus we find

$$\phi(D)xu = x\phi(D)u + \phi'(D)u$$

$$\phi(D)x\phi(D)xu = x^2\phi(D)^2u + 3x\phi(D)\phi'(D)u + \phi(D)\phi''(D)u + \phi'(D)^2u.$$

But this reduction would lead to resulting equations of considerable length, unless we give to $\Phi(D)$ a particular form. This form should be such that negative powers of D may disappear. Or we may make $u = \Phi(D)z$, and so take away such powers, and at the same time may introduce more arbitrary constants into the equation. Thus we should find integrable equations of the second order, having coefficients of the form $a + bx + cx^2$, of considerable generality, owing to the large number of constants which they would contain. The method will be found on trial well adapted to the integration of such equations.

By changing $k$ into $2k$, we shall have

$$\tau^{l-2k} = qD + r,\ \phi(D) = -2k\left(D + \frac{r}{q}\right),\ \lambda(D) = -2k\left(D + \frac{r}{q}\right)\left(m + \frac{\Phi'(D)}{\Phi(D)}\right),$$

$$\varpi' = -2k\left(D + \frac{r}{q}\right)\left(x + m + \frac{\Phi'(D)}{\Phi(D)}\right).$$

With these values we derive from (14.),

$$(\varpi' + nk)(\varpi' + nk + k)u + p\varpi'(qD + r)u = X \quad \text{. . . . . . . . . . .} \quad (17.)$$

$$u = P'\begin{Bmatrix}(n+1)k \\ 2k\end{Bmatrix}^{-1}\{\varpi' + pqD + pr + k\}^{-1}P'\begin{Bmatrix}(n-1)k \\ k\end{Bmatrix}X.$$

Other examples might be given under this head, but I shall now proceed to the solution of two equations somewhat similar to some of those which have been given, but which cannot be solved in the same manner.

## III. THIRD AND FOURTH GENERAL THEOREMS.

Make

$$\varrho = \varepsilon^{\int \frac{\lambda}{\phi} dx},\ \pi_m = \phi D + \Psi + m\lambda,$$

and consequently

$$\pi_n = \phi D + \Psi + n\lambda.$$

We easily verify the equation

$$\varrho^n \pi_{m+n} u = \pi_m \varrho^n u$$

by substituting for $\varrho$ and $\pi_{m+n}$, $\pi_m$ their values, and performing the operations indicated in the result. Therefore by the process followed in the investigation of (*a.*), we find

$$\varrho^n f(\pi_{m+n})u = f(\pi_m)\varrho^n u \quad \text{. . . . . . . . . . . .} \quad (c.)$$

To establish the other general theorem, we make

$$\varrho' = \varepsilon^{\int \frac{\lambda(D)}{\phi(D)} dD},\quad \pi'_m = \phi(D)x + \Psi(D) + m\lambda(D),$$

and therefore also

$$\pi'_n = \phi(D)x + \Psi(D) + n\lambda(D).$$

But we have found

$$\varepsilon^{nf(\mathrm{D})}=\{1+n\mathrm{D}_2 f'(\mathrm{D}_1)+\}\varepsilon^{nf(\mathrm{D}_1)}.$$

Operating with both members of this on

$$\{\phi(\mathrm{D})x+\Psi(\mathrm{D})+m\lambda(\mathrm{D})\}u,$$

we find

$$\varepsilon^{nf(\mathrm{D})}\{\phi(\mathrm{D})x+\Psi(\mathrm{D})+m\lambda(\mathrm{D})\}u=\{\phi(\mathrm{D})x+\Psi(\mathrm{D})+m\lambda(\mathrm{D})+n\phi(\mathrm{D})f'(\mathrm{D})\}\varepsilon^{nf(\mathrm{D})}u,$$

dropping the mark under D as no longer needed.

Make

$$f(\mathrm{D})=\int\frac{\lambda(\mathrm{D})}{\phi(\mathrm{D})}d\mathrm{D},$$

then

$$f'(\mathrm{D})=\frac{\lambda(\mathrm{D})}{\phi(\mathrm{D})},\ \phi(\mathrm{D})f'(\mathrm{D})=\lambda(\mathrm{D}).$$

Substituting this value of $f'(\mathrm{D})$ in the last equation, the result will be equivalent to

$$\varrho'^n\pi'_m u=\pi'_{m+n}\varrho'^n u.$$

Change $u$ into $\varrho'^{-n}u$, and transpose the members, and we have

$$\pi'_{m+n}u=\varrho'^n\pi'_m\varrho'^{-n}u.$$

Therefore, as before, we shall have

$$\varrho'^{-n}f(\pi'_{m+n})u=f(\pi'_m)\varrho'^{-n}u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d.)$$

By the interchange of the symbols $x$ and D, the two general theorems ($c$.) and ($d$.) may be converted the one into the other, $\varepsilon$ at the same time being changed into $\varepsilon^{-1}$.

Before we can employ these theorems in the way intended, we must find the relations between the arbitrary functions required in order that two others may subsist. The first of which is

$$\pi_m\pi_n u=\pi_n\pi_m u+(n-m)a\varrho u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e.)$$

With the values given of $\pi_m$ and $\pi_n$, we easily find that

$$\pi_m\pi_n u=\pi_n\pi_m u+(n-m)\phi\lambda' u.$$

Therefore we must have

$$a\varrho=\phi\lambda',\ \text{or}\ a\varepsilon^{\int\frac{\lambda}{\phi}dx}=\phi\lambda'.$$

Passing to the logarithms of both members,

$$\log a+\int\frac{\lambda}{\phi}dx=\log(\phi\lambda'),$$

and by differentiation

$$\frac{\lambda}{\phi}dx=\frac{d(\phi\lambda')}{\phi\lambda'}.$$

Hence we find successively

$$\lambda\lambda' dx=d(\phi\lambda'),\ \frac{1}{2}d(\lambda^2)=d(\phi\lambda'),\ b+\tfrac{1}{2}\lambda^2=\phi\lambda',\ \text{and}\ \frac{1}{\phi}=\frac{\lambda'}{b+\frac{1}{2}\lambda^2},$$

which is the required relation or condition.

From the last we find

$$\int\frac{\lambda}{\phi}dx=\int\frac{\lambda d\lambda}{b+\frac{1}{2}\lambda^2}=\log\left(b+\frac{1}{2}\lambda^2\right)+\log c.$$

We easily find $c=\frac{1}{a}$, therefore $\varrho=\frac{b+\frac{1}{2}\lambda^2}{a}$.

The other condition alluded to is

$$\pi'_m\pi'_n u=\pi'_n\pi'_m u+(n-m)a\varrho'^{-1}u. \quad \text{(f.)}$$

From the values of $\pi'_m$, $\pi'_n$, we find

$$\pi'_m\pi'_n u=\pi'_n\pi'_m u+(m-n)\phi(\mathrm{D})\lambda'(\mathrm{D}).$$

Therefore we must have

$$-a\varrho'^{-1}=\phi(\mathrm{D})\lambda'(\mathrm{D}),\text{ or }-a\varepsilon^{-\int\frac{\lambda(\mathrm{D})}{\phi(\mathrm{D})}d\mathrm{D}}=\phi(\mathrm{D})\lambda'(\mathrm{D});$$

and as in the former case,

$$-\frac{\lambda(\mathrm{D})}{\phi(\mathrm{D})}d\mathrm{D}=\frac{d\{\phi(\mathrm{D})\lambda'(\mathrm{D})\}}{\phi(\mathrm{D})\lambda'(\mathrm{D})},\ -\lambda(\mathrm{D})d\lambda(\mathrm{D})=d\{\phi(\mathrm{D})\lambda'(\mathrm{D})\}.$$

Therefore, also,

$$b-\frac{1}{2}\lambda(\mathrm{D})^2=\phi(\mathrm{D})\lambda'(\mathrm{D}),\text{ and }\frac{1}{\phi(\mathrm{D})}=\frac{\lambda'(\mathrm{D})}{b-\frac{1}{2}\lambda(\mathrm{D})^2},$$

which is the required relation between $\phi(\mathrm{D})$ and $\lambda(\mathrm{D})$.

Also

$$-\int\frac{\lambda(\mathrm{D})}{\phi(\mathrm{D})}d\mathrm{D}=\log\left(\frac{b-\frac{1}{2}\lambda(\mathrm{D})^2}{c}\right),$$

where we easily perceive that $c=-a$. Whence $\varrho'=\frac{a}{\frac{1}{2}\lambda(\mathrm{D})^2-b}$.

*Application of the four last Theorems and Formulæ.*

Let

$$\pi_m\pi_n u+p\varrho u=\mathrm{X}. \quad (18.)$$

A single case of this equation was solved by Mr. BOOLE in No. 7, New Series of the Cambridge Mathematical Journal, the quantities $\phi$ and $\lambda$ being given functions of the independent variable, and having a given relation, $\Psi$ being nothing. In the Philosophical Magazine, vol. xxxii. p. 257, I gave two similar solutions, but contrived to introduce an arbitrary function of $x$, by which the solution was very greatly extended. Here that method is superseded and the solution rendered much more general. To solve the above, make $u=\pi_{m+1}u_1$; then

$$\pi_m\pi_n\pi_{m+1}u_1+p\varrho\pi_{m+1}u_1=\mathrm{X}\text{ by substitution,}$$

$$\pi_m\pi_n\pi_{m+1}u_1+p\pi_m\varrho u_1=\mathrm{X}\text{ by (c.),}$$

$$\pi_n\pi_{m+1}u_1+p\varrho u_1=\pi_m^{-1}\mathrm{X},$$

$$\pi_{m+1}\pi_n u_1+p_1\varrho u_1=\pi_m^{-1}\mathrm{X}\text{ by (e.), }p_1=p-a(n-m-1).$$

After $i$ transformations, we have

$$\pi_{m+i}\pi_n u_i + p_i \varrho u_1 = \pi_{m+i-1}^{-1} \ldots\ldots \pi_m^{-1} \mathrm{X}.$$

If $p_i=0$, or $p=ia\left(n-m-\frac{i+1}{2}\right)$, the equation is solved, and we have

$$u_i = \pi_n^{-1}\pi_{m+i}^{-1} \ldots\ldots \pi_m^{-1}\mathrm{X}.$$

From this we easily find $u$. The success of the method, it will be seen, depends upon the value of $p$.

Other cases and solutions may be seen in the Philosophical Magazine in the article before mentioned, but they cannot be given here.

Reduced to the ordinary form, (18.) becomes

$$\varphi^2\mathrm{D}^2u + \varphi(\varphi' + 2\Psi + (m+n)\lambda)\mathrm{D}u + (\varphi(\varphi' + n\lambda') + (\Psi + m\lambda)(\Psi + n\lambda) + p\varrho)u = \mathrm{X}.$$

If we were to substitute for $\varphi$ and $\varrho$ their values, this would become very complicated. We see that it differs considerably in form from the preceding examples which have been thus reduced.

There is one equation bearing some analogy to that which has just been solved which deserves to be noticed here, although its solution must be effected by a very different process. It is

$$\pi_m\pi_n u + p\varrho^{2(n-m)}u = \mathrm{X}. \quad \ldots\ldots\ldots\ldots \quad (19.)$$

By ($c$.) this may be put under the form

$$\pi_m\varrho^{m-n}\pi_m\varrho^{n-m}u + p\varrho^{2(n-m)}u = \mathrm{X}.$$

Make $u=\varrho^{2(m-n)}v$, and the last, by substituting this value, will become

$$\pi_m\varrho^{m-n}\pi_m\varrho^{m-n}v + pv = \mathrm{X}.$$

Now let $\pi_m\varrho^{m-n}=\tau$, and the preceding will be reduced to

$$(\tau^2+p)v = \mathrm{X}.$$

Whence

$$v = (\tau^2+p)^{-1}\mathrm{X} = \frac{\mathrm{A}}{\tau - p^{\frac{1}{2}}\sqrt{-1}}\mathrm{X} - \frac{\mathrm{A}}{\tau + p^{\frac{1}{2}}\sqrt{-1}}\mathrm{X},$$

where

$$\mathrm{A} = \frac{1}{2p^{\frac{1}{2}}\sqrt{-1}}.$$

If $v=v_1+v_2$, we may evidently make

$$(\tau - p^{\frac{1}{2}}\sqrt{-1})v_1 = \mathrm{AX}, \quad (\tau + p^{\frac{1}{2}}\sqrt{-1})v_2 = -\mathrm{AX};$$

for these lead to the same result. And thus $v$ is found from two equations of a lower order than the given equation.

If we now put for $\tau$ its value in the two last equations, we shall have

$$\pi_m\varrho^{m-n}v_1 - p^{\frac{1}{2}}\sqrt{-1}v_1 = \mathrm{AX}, \quad \pi_m\varrho^{m-n}v_2 + p^{\frac{1}{2}}\sqrt{-1}v_2 = -\mathrm{AX}.$$

If we put $\varrho^{m-n}v_1=\omega_1$, $\varrho^{m-n}v_2=\omega_2$, these will take the more convenient form

$$\pi_m\omega_1-p^{\frac{1}{2}}\sqrt{-1}\varrho^{n-m}\omega_1=\mathrm{AX},\quad \pi_m\omega_2+p^{\frac{1}{2}}\sqrt{-1}\varrho^{n-m}\omega_2=-\mathrm{AX};$$

which are only of the first order, since $\pi_m$ is of that order. The equation (19.) may therefore be considered as integrated.

By the conversion of symbols (18.) and (19.) will be changed into

$$\left.\begin{array}{l}\pi'_m\pi'_n u+p\varrho'^{-1}u=\mathrm{X},\\ \pi'_m\pi'_n u+p\varrho'^{2(m-n)}u=\mathrm{X}\end{array}\right\} \quad \ldots \ldots \quad (20.)$$

These may be solved by means of the formulæ (*d.*) and (*f.*) in exactly the same way as (18.) and (19.), and every step of the solution in the one case may be derived from the corresponding step of the solution in the other merely by the conversion of symbols. But every solution of these equations will not be a practicable one, or be susceptible of interpretation in finite terms. The operations $\pi'^{-1}_{m+r}$ however can be performed if

$$\varepsilon^{\int\left(\frac{\Psi(\mathrm{D})+(m+r)\lambda(\mathrm{D})}{\varphi(\mathrm{D})}\right)d\mathrm{D}}=\chi(\mathrm{D})\varepsilon^{q\mathrm{D}},$$

$\chi(\mathrm{D})$ being a rational function of D, and the constant $q$ being positive, negative, or nothing. By differentiating the logarithms of each member relative to D, this will give

$$\frac{\Psi(\mathrm{D})+(m+r)\lambda(\mathrm{D})}{\varphi(\mathrm{D})}=\frac{q\chi(\mathrm{D})+\chi'(\mathrm{D})}{\chi(\mathrm{D})},$$

or

$$\frac{\Psi(\mathrm{D})+(m+r)\lambda(\mathrm{D})}{b-\frac{1}{2}\lambda(\mathrm{D})^2}=\frac{q\chi(\mathrm{D})+\chi'(\mathrm{D})}{\chi(\mathrm{D})\lambda'(\mathrm{D})},$$

by putting for $\frac{1}{\varphi(\mathrm{D})}$ its value, and dividing the equation by $\lambda'(\mathrm{D})$. Therefore

$$\Psi(\mathrm{D})=\frac{q\chi(\mathrm{D})+\chi'(\mathrm{D})}{\chi(\mathrm{D})\lambda'(\mathrm{D})}\left(b-\tfrac{1}{2}\lambda(\mathrm{D})^2\right)-(m+r)\lambda(\mathrm{D}).$$

Such is the value which $\Psi(\mathrm{D})$ must have in order that the solution may be practicable, $\lambda(\mathrm{D})$ being at the same time a rational function of D.

The value of $\Psi(\mathrm{D})$ is too complex for practical utility. But if we make

$$b=\tfrac{1}{2}c^2,\quad \lambda(\mathrm{D})=a\mathrm{D}+c,$$

we shall have

$$\frac{b-\frac{1}{2}\lambda(\mathrm{D})^2}{\lambda'(\mathrm{D})}=-\tfrac{1}{2}\mathrm{D}(a\mathrm{D}+2c).$$

If therefore $\chi(\mathrm{D})=a\mathrm{D}+2c$, then we find

$$\Psi(\mathrm{D})=-\tfrac{1}{2}aq\mathrm{D}^2-\left(am+ar+cq+\tfrac{1}{2}a\right)\mathrm{D}-(cm+cr).$$

But if $\chi(\mathrm{D})=\frac{1}{2}\mathrm{D}$, we shall have

$$\Psi(\mathrm{D})=-\frac{1}{2}aq\mathrm{D}^2-\left(am+ar+cq+\frac{1}{2}a\right)\mathrm{D}-(cm+cr+c).$$

With either of these values, and by suitably assuming $m$ and $n$, we may find convenient practicable forms for both the equations (20.). And perhaps we may succeed with other assumptions. If we admit complex forms, we may have them in abundance.

## IV. SOLUTION OR REDUCTION OF ANOTHER SERIES OF EQUATIONS.

The equations here treated of are generalizations and extensions of one solved by Mr. BOOLE in the Cambridge Mathematical Journal, No. 1, New Series. In that equation the coefficients are integer functions of $x$, here they may be any functions of that quantity consistent with the conditions of integrability. Also the symbol D is replaced by $\varpi$, and arbitrary functions of this last are introduced.

In order to abridge I shall write $\varpi_a$ for $\varpi+a$, and $f(\varpi)$ will denote any rational function of $\varpi$. This being premised, let

$$f(\varpi)f(\varpi-k)\varpi(\varpi+k)u+p\varpi_a(\varpi_a+(2n+1)k)\tau^{2k}u=\mathrm{X}. \quad \ldots \quad (21.)$$

Assume

$$u=(\varpi_a+3k)(\varpi_a+5k)\ldots.(\varpi_a+(2n+1)k)v,$$

the common difference of the factors being here $2k$. Then by ($a$.) we shall have

$$(\varpi_a+(2n+1)k)\tau^{2k}u=(\varpi_a+k)\ldots.(\varpi_a+(2n+1)k)\tau^{2k}v.$$

Substituting these values and reducing the result, we find

$$f(\varpi)f(\varpi-k)\varpi(\varpi+k)v+p\varpi_a(\varpi_a+k)\tau^{2k}v=\mathrm{P}\begin{Bmatrix}(2n+1)k\\3k\end{Bmatrix}\mathrm{X}_1,$$

where the $a$ is suppressed in the factorial of the second member for brevity. If we put

$$f(\varpi)^{-1}f(\varpi-k)^{-1}\varpi^{-1}(\varpi+k)^{-1}\mathrm{P}\begin{Bmatrix}(2n+1)k\\3k\end{Bmatrix}^{-1}\mathrm{X}=\mathrm{X}_1,$$

we have

$$v+p\frac{(\varpi_a+k)\varpi_a}{f(\varpi)(\varpi+k)f(\varpi-k)\varpi}\tau^{2k}v=\mathrm{X}_1,$$

or

$$v+p\left(\frac{\varpi_a+k}{f(\varpi)(\varpi+k)}\right)\tau^k\left(\frac{\varpi_a+k}{f(\varpi)(\varpi+k)}\right)\tau^k v=\mathrm{X}_1, \text{ by } (a.).$$

Make

$$\left(\frac{\varpi_a+k}{f(\varpi)(\varpi+k)}\right)\tau^k=\varrho,$$

and the last becomes

$$v+p\varrho^2v=\mathrm{X}_1.$$

The mode of treating this has been already explained. By it the proposed is made to depend upon two others, each of which is of an order only half as high. In certain cases therefore this reduction amounts to a solution.

Let us next take the equation

$$\varpi_a(\varpi_a+(2n+1)k)u+pf(\varpi)f(\varpi-k)\varpi(\varpi+k)\tau^{2k}u=X, \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

the arbitrary functions being put in the second term, for this only amounts to changing them and X.

Assume

$$u=(\varpi_a+3k)^{-1}\ldots\ldots(\varpi_a+(2n+1)k)^{-1}v.$$

Proceeding exactly as heretofore, we find

$$v+p\frac{f(\varpi)(\varpi+k)f(\varpi-k)\varpi}{(\varpi_a+k)\varpi_a}\tau^{2k}v=X_1,$$

if

$$X_1=\varpi_a^{-1}(\varpi_a+k)^{-1}P\begin{Bmatrix}(2n-1)k\\ k\end{Bmatrix}X.$$

Making

$$\left(\frac{f(\varpi)(\varpi+k)}{\varpi_a+k}\right)\tau^k=\varrho,$$

by (*a.*) the last becomes

$$v+p\varrho^2v=X_1.$$

In the next two examples we shall put the equations under a somewhat different form, on account of their complexity. Suppose

$$u+\frac{\varpi_a(\varpi_a+k)(\varpi_a+(3n+2)k)}{f(\varpi)f(\varpi-k)f(\varpi-2k)\varpi(\varpi+k)(\varpi+2k)}p\tau^{3k}u=X.$$

Here we assume

$$u=(\varpi_a+5k)\ldots.(\varpi_a+(3n+2)k)v,$$

the common difference of the factors being $3k$.

If

$$X_1=P\begin{Bmatrix}(3n+2)k\\ 5k\end{Bmatrix}^{-1}X,\quad \varrho=\left(\frac{\varpi_a+2k}{f(\varpi)(\varpi+2k)}\right)\tau^k,$$

we shall have in this case

$$v+p\varrho^3v=X_1,$$

which may be replaced by three equations, each containing the first power only of $\varrho$.

As a last example let

$$u+\frac{f(\varpi)f(\varpi-k)f(\varpi-2k)\varpi(\varpi+k)(\varpi+2k)}{\varpi_a(\varpi_a+k)(\varpi_a+(3n+2)k)}p\tau^{3k}u=X.$$

Here

$$u=(\varpi_a+5k)^{-1}\ldots.(\varpi_a+(3n+2)k)^{-1}v,\quad \varrho=\left(\frac{f(\varpi)(\varpi+2k)}{\varpi_a+2k}\right)\tau^k,$$

and also

$$v+p\varrho^3v=X_1.$$

This series of equations may be continued at pleasure; and it is obvious that in the whole series, if we change $\varpi$ into $\varpi'$, and $\tau$ into $\tau'^{-1}$, the resulting equations may be reduced by (*b.*) exactly as these have been by (*a.*), and that the solutions or reductions of the one series may be obtained from those of the other by the conversion of symbols.

Mr. BOOLE's general equation

$$X=u+af(D)\varepsilon^{\theta}u+bf(D)f(D-k)\varepsilon^{2\theta}u+\dots..$$

may be generalized and extended in the same way. Thus if

$$X=u+af(\varpi)\tau^{k}u+bf(\varpi)f(\varpi-k)\tau^{2k}u+\dots..,$$

or

$$X=u+af(\varpi')\tau'^{-k}u+bf(\varpi')f(\varpi'-k)\tau'^{-2k}u+\dots.,$$

by making $\varrho=f(\varpi)\tau^{k}$ in the first, and $\varrho'=f(\varpi')\tau'^{-k}$ in the second, we shall have

$$X=u+a\varrho u+b\varrho^{2}u+\dots=(1-p_1\varrho)(1-p_2\varrho)\dots.u,$$

$$X=u+a\varrho'u+b\varrho'^{2}u+\dots.=(1-p_1\varrho')(1-p_2\varrho')\dots.u.$$

And thus each equation, by the method explained further back, will be reduced to a number of others, each of which is much more simple than the proposed.

It must be observed, that the method which has been applied to the solution of each particular class of equations will not apply to either of the other classes. We see the same thing when employing other methods, and we see no reason to suppose that our means of integrating equations will ever be greatly extended otherwise than by the multiplication or aggregation of particular methods. Such methods therefore ought not to be considered as possessing little interest. The same thing may be inferred from the various conditions of integrability at which we arrive in those cases where we can treat the same sort of equations by different methods.

Some of the examples which have been given in this paper, when reduced to the ordinary form, are very complex; but when particular forms are assigned to the arbitrary functions, results sufficiently simple may be obtained. If we will accept none but such as at first sight present themselves under a very simple form, we must not expect any great extension of our present scanty means of integration; for these equations are usually very easily integrated. The following example may serve as an illustration.

The theorem

$$\phi D^{n}u=D^{n}\phi u-nD^{n-1}\phi'u+\frac{n(n-1)}{2}D^{n-2}\phi''u-\text{ \&c.}$$

is easily verified by performing the operations $D^{n}$, $D^{n-1}$, &c. in the second member by means of the formula

$$D^{n}=D_1^{n}+nD_2D_1^{n-1}+\frac{n(n-1)}{2}D_2^{2}D_1^{n-2}+\text{ \&c.}$$

Now let there be given the equation

$$X=\Psi u+\Psi_1Du+\Psi_2D^{2}u+\Psi_3D^{3}u+\dots..$$

and let this, by the preceding theorem, be transformed into

$$X=\Phi u+D\Phi_1u+D^{2}\Phi_2u+D^{3}\Phi_3u+\dots..$$

We shall have

$$\Phi=\Psi-\Psi_1'+\Psi_2''-\Psi_3'''+\dots$$

$$\Phi_1=\Psi_1-2\Psi_2'+3\Psi_3''-\dots.$$

$$\Phi_2=\Psi_2-3\Psi_3'+\dots..$$

$$\Phi_3=\Psi_3-\dots..\text{ \&c.}$$

If $\Phi=0$, the equation is integrable once, and we shall have

$$D^{-1}X=\Phi_1 u+D\Phi_2 u+D^2\Phi_3 u+\ldots.$$

If both $\Phi=0$, and $\Phi_1=0$, it is integrable twice, and we have

$$D^{-2}X=\Phi_2 u+D\Phi_3 u+\ldots.$$

Let us take as an example the following equation of the second order,

$$\Psi_2 D^2 u+\Psi_1 Du+(\Psi'_1-\Psi''_2)u=X.$$

The transformed gives

$$D\Phi_2 u+\Phi_1 u=D^{-1}X,$$

or

$$\Psi_2 Du+(\Psi_1-\Psi'_2)u=D^{-1}X,$$

and

$$u=\left(D+\frac{\Psi_1-\Psi'_2}{\Psi_2}\right)^{-1}\Psi_2^{-1}D^{-1}X=\Psi_2\varepsilon^{-\int\frac{\Psi_1}{\Psi_2}dx}D^{-1}\Psi_2^{-2}\varepsilon^{\int\frac{\Psi_1}{\Psi_2}dx}D^{-1}X.$$

This example was given by Mr. HARGREAVE in the Philosophical Transactions, 1848.

But the proposed equation may at once be put under the form

$$D\{\Psi_2 Du+(\Psi_1-\Psi'_2)u\}=X,$$

which is immediately integrable, giving

$$\Psi_2 Du+(\Psi_1-\Psi'_2)u=D^{-1}X,$$

as before found, but this is far from being a solitary example.

## V. FIFTH AND SIXTH GENERAL THEOREMS.

The theorems now to be given bear no resemblance to those which have been heretofore investigated, nor is their mode of application at all similar. Making

$$\phi\chi'=\chi_1,\ \phi\chi'_1=\chi_2,\ \phi\chi'_2=\chi_3,\ \&c.,$$

$$\chi\varpi^n u=\varpi^n\chi u-n\varpi^{n-1}\chi_1 u+\frac{n(n-1)}{2}\varpi^{n-2}\chi_2 u-\&c. \quad \ldots \quad (g.)$$

will be verified by putting for $\varpi$ its value $\phi D+\lambda$, and actually performing the operations denoted by D, which may be readily done by the method which has now been repeatedly explained.

Again, making

$$\phi(D)\chi'(D)=\chi_1(D),\ \phi(D)\chi'_1(D)=\chi_2(D),\ \phi(D)\chi'_2(D)=\chi_3(D),\ \&c.,$$

we shall have

$$\chi(D)\varpi'^n u=\varpi'^n\chi(D)u+n\varpi'^{n-1}\chi_1(D)u+\frac{n(n-1)}{2}\varpi'^{n-2}\chi_2(D)u+\&c. \quad \ldots \quad (h.)$$

This also may be verified by putting for $\varpi'$ its value $\phi(D)x+\lambda(D)$, and performing the operations D, where required, by the formula

$$f(D)=f(D_1)+D_2 f'(D_1)+\tfrac{1}{2}D_2^2 f''(D_1)+\&c.,$$

and afterwards dropping the marks of distinction.

*Application of the Theorem* (g.) *to Integration.*

To apply the theorem (*g*.), let us suppose that we have given the equation

$$X = \Psi u + \Psi^1 \varpi u + \Psi^2 \varpi^2 u + \&c. \quad \ldots \ldots \quad (23.)$$

Let this by (*g*.) be transformed into

$$X = \Phi u + \varpi \Phi_1 u + \varpi^2 \Phi_2 u + \&c.$$

We shall have

$$\Phi = \Psi - \Psi^1_1 + \Psi^2_2 - \Psi^3_2 + \ldots$$

$$\Phi_1 = \Psi^1 - 2\Psi^2_1 + 3\Psi^3_2 - \ldots.$$

$$\Phi_2 = \Psi^2 - 3\Psi^3_1 + \ldots..$$

$$\Phi_3 = \Psi^3 - \ldots. \&c.$$

The figure at the top of $\Psi$ denotes the place of the term in the given equation, that at the bottom marks the term in (*g*.) corresponding to it*. By going a little into the operations, the meaning of these symbols will be easily understood.

If $\Phi = 0$, or $\Psi = \Psi^1_1 - \Psi^2_2 + \ldots$ the equation thus transformed will be integrable once, and we shall have

$$\varpi^{-1} X = \Phi_1 u + \varpi \Phi_2 u + \varpi^2 \Phi_3 u + \&c.$$

If, moreover, $\Phi_1 = 0$, it will be twice integrable.

Let us take as a more particular example the equation

$$\Psi^2 \varpi^2 u + \Psi^1 \varpi u + (\Psi^1_1 - \Psi^2_2) u = X,$$

which is integrable because $\Phi = 0$. Putting for $\Psi^1_1$, $\Psi^2_2$ their values, this becomes

$$\Psi^2 \varpi^2 u + \Psi^1 \varpi u + \phi(\Psi^{1\prime} - \phi' \Psi^{2\prime} - \phi \Psi^{2\prime\prime}) u = X. \quad \ldots \ldots \quad (24.)$$

The transformed equation will be

$$\varpi^2 \Phi_2 u + \varpi \Phi_1 u = X,$$

or

$$\varpi \Phi_2 u + \Phi_1 u = \varpi^{-1} X;$$

which by substituting for $\varpi$ its value, is easily reduced to

$$Du + \left(\frac{\Phi_2'}{\Phi_2} + \frac{\lambda}{\phi} + \frac{\Phi_1}{\phi \Phi_2}\right) u = (\phi \Phi_2)^{-1} \varpi^{-1} X.$$

In order to abridge, make

$$\frac{\Phi_2'}{\Phi_2} + \frac{\lambda}{\phi} + \frac{\Phi_1}{\phi \Phi_2} = \frac{\lambda}{\phi} + \frac{\Psi^1}{\phi \Psi^2} - \frac{\Psi^{2\prime}}{\Psi^2} = \theta;$$

then the last becomes

$$(D + \theta) u = (\phi \Psi^2)^{-1} \varpi^{-1} X,$$

which gives

$$u = (D + \theta)^{-1} (\phi \Psi^2)^{-1} \varpi^{-1} X,$$

or

$$u = \varepsilon^{-\int \theta dx} D^{-1} \varepsilon^{\int \theta dx} (\phi \Psi^2)^{-1} \varepsilon^{-\int \frac{\lambda}{\phi} dx} D^{-1} \varepsilon^{\int \frac{\lambda}{\phi} dx} \phi^{-1} X.$$

* The figures at the top of $\Psi$ are not exponents of powers, but distinct functional marks.

If (24.) be reduced to the ordinary form, it becomes

$$D^2u+\left(\frac{\phi'+2\lambda}{\phi}+\frac{\Psi^1}{\phi\Psi^2}\right)Du+\left(\frac{\phi\lambda'+\lambda^2}{\phi^2}+\frac{\Psi'(\phi+\lambda)}{\phi^2\Psi^2}+\frac{\Psi^{1\prime}-\phi'\Psi^{2\prime}-\phi\Psi^{2\prime\prime}}{\phi\Psi^2}\right)u=\frac{X}{\phi^2\Psi^2}.$$

This would give a very large number of integrable equations by assigning particular forms to the arbitrary functions $\phi$, $\lambda$, $\Psi^1$, and $\Psi^2$. And indeed every one of the equations treated of in this paper, by giving particular forms to the arbitrary functions, would furnish a large number of particular ones. This circumstance makes the chance of our being able to put a given equation under some one of these forms the greater, and consequently in this respect enhances the value of the methods employed.

*Application of the Theorem* (h.).

The equations to be reduced by (*h.*) are of the form

$$X=\Psi(D)u+\Psi^1(D)\varpi'u+\Psi^2(D)\varpi'^2u+\&c. \quad \dots\dots\dots \quad (25.)$$

This is deduced from (23.) by the interchange of the symbols D and $x$, but the theorems (*g.*) and (*h.*) by which the transformations are effected cannot be thus deduced the one from the other.

The above will be transformed by (*h.*) into

$$X=\Phi(D)u+\varpi'\Phi_1(D)u+\varpi'^2\Phi_2(D)u+\dots,$$

where

$$\Phi(D)=\Psi(D)+\Psi_1^1(D)+\Psi_2^2(D)+\Psi_3^3(D)+\dots$$

$$\Phi_1(D)=\Psi^1(D)+2\Psi_1^2(D)+3\Psi_2^3(D)+\dots$$

$$\Phi_2(D)=\Psi^2(D)+3\Psi_1^3(D)+\dots$$

$$\Phi_3(D)=\Psi^3(D)+\dots\&c.$$

If

$$\Phi(D)=\Psi(D)+\Psi_1^1(D)+\Psi_2^2(D)+\dots=0,$$

or

$$\Psi(D)=-\Psi_1^1(D)-\Psi_2^2(D)-\dots,$$

we shall have

$$\varpi'^{-1}X=\Phi_1(D)u+\varpi'\Phi_2(D)u+\dots$$

We might take as examples equations of a higher order than the second, but as these last are of the greater importance I have hitherto selected them, and shall take here the equation

$$\Psi^2(D)\varpi'^2u+\Psi^1(D)\varpi'u-\left(\Psi_1^1(D)+\Psi_2^2(D)\right)u=X,$$

which by putting for $\Psi_1^1(D)$ and $\Psi_2^2(D)$ their values, the figures here at the top and bottom of $\Psi$ signifying the same as in the former case, becomes

$$\Psi^2(D)\varpi'^2u+\Psi^1(D)\varpi'u-\phi(D)\{\Psi^{1\prime}(D)+\phi'(D)\Psi^{2\prime}(D)+\phi(D)\Psi^{2\prime\prime}(D)\}u=X. \quad (26.)$$

The transformed equation gives

$$\varpi'\Phi_2(D)u+\Phi_1(D)u=\varpi'^{-1}X;$$

which, reduced to the most simple form, is equivalent to

$$\phi(D)\Phi_2(D)xu+\{\Phi_1(D)+\lambda(D)\Phi_2(D)-\phi(D)\Phi_2'(D)\}u=\varpi'^{-1}X.$$

Make

$$\theta(D)=\frac{\Phi_1(D)}{\phi(D)\Phi_2(D)}+\frac{\lambda(D)}{\phi(D)}-\frac{\Phi_2'(D)}{\Phi_2(D)}$$
$$=\frac{\Psi^1(D)}{\phi(D)\Psi^2(D)}+\frac{\lambda(D)}{\phi(D)}+\frac{\Psi^{2\prime}(D)}{\Psi^2(D)}.$$

Substituting this value in the last equation, after dividing both members by $\phi(D)\Phi_2(D)$, or in other words, operating with $\phi(D)^{-1}\Phi_2(D)^{-1}$ on both members, we have

$$\{x+\theta(D)\}u=\{\phi(D)\Phi_2(D)\}^{-1}\varpi'^{-1}X$$

and

$$u=\{x+\theta(D)\}^{-1}\{\phi(D)\Psi^2(D)\}^{-1}\varpi'^{-1}X,$$

where the value of $\Phi_2(D)$ has been substituted. By further reduction,

$$u=\varepsilon^{\int\theta(D)dD}x^{-1}\varepsilon^{-\int\theta(D)dD}\big(\phi(D)\Psi^2(D)\big)^{-1}\varepsilon^{\int\frac{\lambda(D)}{\phi(D)}dD}x^{-1}\varepsilon^{-\int\frac{\lambda(D)}{\phi(D)}dD}\phi(D)^{-1}X.$$

The equation itself requires that $\Psi^1(D)$, $\Psi^2(D)$, $\phi(D)$, and $\lambda(D)$ should be rational functions of (D); and that the solution may be practicable, we must have

$$\varepsilon^{\int\frac{\lambda(D)}{\phi(D)}dD}=\chi(D)\varepsilon^{mD},$$

$\chi(D)$ being a rational function. Also the term $\frac{\Psi^1(D)}{\phi(D)\Psi^2(D)}$ in the value of $\theta(D)$ requires that we should have

$$\varepsilon^{\int\frac{\Psi^1(D)dD}{\phi(D)\Psi^2(D)}}=\Phi(D)\varepsilon^{pD},$$

$\Phi(D)$ being a rational function. These assumptions give, by taking the differentials of their logarithms,

$$\lambda(D)=\phi(D)\Big(m+\frac{\chi'(D)}{\chi(D)}\Big),\ \Psi^1(D)=\phi(D)\Psi^2(D)\Big(p+\frac{\Phi'(D)}{\Phi(D)}\Big).$$

If we were to substitute these values in (26.), we should have a very complex resulting equation; but by giving suitable particular values to the arbitrary functions, and perhaps by changing $u$ into $f(D)z$, giving to $f(D)$ a convenient form, we might obtain resulting equations sufficiently simple, and we might obtain some very general of their kind, remembering to introduce as many arbitrary constants as we can.

Make

$$\chi(D)=\phi(D),\ \Phi(D)=\Psi^2(D);$$

then

$$\lambda(D)=m\phi(D)+\phi'(D),\ \Psi^1(D)=\phi(D)\big(p\Psi^2(D)+\Psi^{2\prime}(D)\big).$$

With these values (26.) becomes, dropping the figure 2 on the top of $\Psi$, as being no longer needed,

$$\Psi(D)\varpi'^2u+\phi(D)\big(p\Psi(D)+\Psi'(D)\big)\varpi'u-\phi(D)\frac{d}{dD}\Big\{\phi(D)\big(p\Psi(D)+2\Psi'(D)\big)\Big\}u=X, \quad (27.)$$

which is of a remarkable form. Also

$$u=\phi(\mathrm{D})\Psi(\mathrm{D})^2\varepsilon^{(m+p)\mathrm{D}}x^{-1}\{\phi(\mathrm{D})\Psi(\mathrm{D})^3\}^{-1}\varepsilon^{-p\mathrm{D}}x^{-1}\phi(\mathrm{D})^{-2}\varepsilon^{-m\mathrm{D}}\mathrm{X}.$$

The equation (27.) might very well be reduced to the ordinary form without particularizing the arbitrary functions.

If we make

$$\chi(\mathrm{D})=\phi(\mathrm{D}),\ \Phi(\mathrm{D}=\phi(\mathrm{D}),$$

and change $\Psi^2$ into $\Psi$, we shall have

$$\lambda(\mathrm{D})=m\phi(\mathrm{D})+\phi'(\mathrm{D}),\ \Psi^1(\mathrm{D})=\Psi(\mathrm{D})\big(p\phi(\mathrm{D})+\phi'(\mathrm{D})\big).$$

With these values (26.) will become, operating with $\Psi(\mathrm{D})^{-1}$ on both members,

$$\varpi'^2u+\big(p\phi(\mathrm{D})+\phi'(\mathrm{D})\big)\varpi'u-\frac{\phi(\mathrm{D})}{\Psi(\mathrm{D})}\frac{d}{d\mathrm{D}}\Big\{p\phi(\mathrm{D})\Psi(\mathrm{D})+\frac{d}{d\mathrm{D}}\big(\phi(\mathrm{D})\Psi(\mathrm{D})\big)\Big\}u=\Psi(\mathrm{D})^{-1}\mathrm{X}, \quad (28.)$$

which is of a form not a little singular, and which might be put under the common form, retaining the arbitrary functions.

The value of $u$ in this example is

$$u=\phi(\mathrm{D})^2\Psi(\mathrm{D})\varepsilon^{(m+p)\mathrm{D}}x^{-1}\{\phi(\mathrm{D})\Psi(\mathrm{D})\}^{-2}\varepsilon^{-p\mathrm{D}}x^{-1}\phi(\mathrm{D})^{-2}\varepsilon^{-m\mathrm{D}}\mathrm{X}.$$

In the two last examples we see from the expression of the value of $u$ that they are practicable, or that their solutions can be interpreted.

It may be well to observe, that if we make

$$\phi(\mathrm{D})=\alpha(\mathrm{D})\chi(\mathrm{D}),\ \Psi^2(\mathrm{D})=\beta(\mathrm{D})\Phi(\mathrm{D}),$$

which give

$$\lambda(\mathrm{D})=\alpha(\mathrm{D})\big(m\chi(\mathrm{D})+\chi'(\mathrm{D})\big),\ \Psi^1(\mathrm{D})=\alpha(\mathrm{D})\beta(\mathrm{D})\chi(\mathrm{D})\big(p\Phi(\mathrm{D})+\Phi'(\mathrm{D})\big);$$

and if we eliminate both $\phi(\mathrm{D})$ and $\lambda(\mathrm{D})$ from the value of $\varpi'$, and $\Psi^1(\mathrm{D})$, $\Psi^2(\mathrm{D})$ from (26.), our equation by the substitution of these values will then contain the four arbitrary functions $\alpha(\mathrm{D})$, $\beta(\mathrm{D})$, $\chi(\mathrm{D})$ and $\Phi(\mathrm{D})$. These should then be made integer functions of D; but after actual substitution the resulting equation would be very complicated, unless we first give particular forms to the arbitrary functions. If this be done, and as many constants as possible be introduced, particular integrable equations of interest and value, and of sufficient simplicity, may be obtained.

I shall terminate this paper by observing, that in our attempts at integration, we are apt to seek for equations which immediately present themselves under simple forms, and by this means fall chiefly, or altogether, upon such as could previously be integrated.

*Gunthwaite Hall, near Barnsley, Yorkshire,*
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